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Differential Geometry and its Applications 22 (2005) 215–227

DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS

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Piunikhin–Salamon–Schwarz isomorphisms for Lagrangian intersections

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Received 27 June 2003; received in revised form 5 January 2004

Available online 5 November 2004

Communicated by Y. Eliashberg

Abstract

We study the intersections of gradient trajectories and holomorphic discs with Lagrangian boundary conditions in cotangent bundles, and give a construction of Piunikhin–Salamon–Schwarz isomorphisms in Lagrangian intersections Floer homology.

MSC: primary 37J05; secondary 53D12, 53D40

Keywords: Lagrangian submanifolds; Floer homology; Morse theory

1. Introduction

Let L_0, L_1 be two Lagrangian submanifolds of a symplectic manifold P. We will assume that the intersection $L_0 \cap L_1$ is transverse. Floer chain groups $CF_*(L_0, L_1)$ are $\mathbb{Z}_2$-vector spaces generated by the set $L_0 \cap L_1$. Under certain conditions formulated in [2], Floer homology $HF_*(L_0, L_1)$ for the pair L_0, L_1 is defined as the homology group of $CF_*(L_0, L_1)$ with respect to the boundary operator

$$\partial_F : CF_*(L_0, L_1) \to CF_*(L_0, L_1), \quad \partial_F(x) := \sum_{y \in L_0 \cap L_1} n(x, y)y,$$

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doi:10.1016/j.difgeo.2004.10.008

where $n(x, y)$ are the numbers of the solutions of an elliptic system

$$\begin{cases} \frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} = 0, \\ u(s,i) \in L_i,\ i \in \{0,1\}, \\ u(-\infty,t) \equiv x,\ u(+\infty,t) \equiv y,\ x,y \in L_0 \cap L_1 \end{cases} \tag{1}$$

$(\mathrm{mod}\,\mathbb{Z}_2)$. Floer [1] proved that when $P = T^*M$ is a cotangent bundle over a compact manifold M, $L_0 = O_M$ is a zero section, and $L_1 = \phi_1^H(L_0)$ a Hamiltonian deformation of L_0 Floer homology is isomorphic to the singular homology of M. In that case the system (1) is equivalent to

$$\begin{cases} \frac{\partial u}{\partial s} + J(\frac{\partial u}{\partial t} - X_H(u)) = 0, \\ u(s,i) \in L_0,\ i \in \{0,1\}, \\ u(-\infty,t) = \phi_t^H((\phi_1^H)^{-1})(x), \\ u(+\infty,t) = \phi_t^H((\phi_1^H)^{-1})(y),\ x,y \in L_0 \cap L_1. \end{cases} \tag{2}$$

System (2) is the "negative gradient flow" of the Hamiltonian action functional $\mathcal{A}_H(\gamma) = \int_\gamma p\,dq - H\,dt$ restricted to the paths with ends on the zero section. We will denote Floer homology groups $HF_*(O_M, \phi_1^H(O_M))$ by $HF_*(H)$. For two Hamiltonians H^α, H^β the corresponding Floer homology groups are isomorphic. Fix $R_0 > 0$. Let $H^{\alpha\beta}(s,t,x)$ be a smooth function such that $H^{\alpha\beta}(s,t,x) = H^\alpha(t,x)$ for $s \leqslant -R_0$ and $H^{\alpha\beta}(s,t,x) = H^\beta(t,x)$ for $s \geqslant R_0$. The isomorphism

$$S^{\alpha\beta} : HF_*(H^\alpha) \to HF_*(H^\beta), \quad S^{\alpha\beta}(x^\alpha) = \sum_{x^\beta} n(x^\alpha, x^\beta)x^\beta$$

is defined by counting the numbers $n(x^\alpha, x^\beta)$ of the solutions of the system

$$\begin{cases} \frac{\partial u}{\partial s} + J(\frac{\partial u}{\partial t} - X_{H^{\alpha\beta}}(u)) = 0, \\ u(s,i) \in L_0,\ i \in \{0,1\}, \\ u(-\infty,t) = \phi_t^{H^\alpha}((\phi_1^{H^\alpha})^{-1})(x^\alpha),\ u(+\infty,t) = \phi_t^{H^\beta}((\phi_1^{H^\beta})^{-1})(x^\beta), \\ x^\alpha \in O_M \cap \phi_1^{H^\alpha}(O_M),\ x^\beta \in O_M \cap \phi_1^{H^\beta}(O_M). \end{cases} \tag{3}$$

The isomorphisms between Floer and singular homologies is established via Morse theory. Recall that the Morse chain complex of a Morse function $f : M \to \mathbb{R}$ is a $\mathbb{Z}_2$-vector space $CM_*(f)$ generated by the set of critical points of f. Morse homology groups $HM_*(f)$ are the homology groups of $CM_*(f)$ with respect to the boundary operator

$$\partial_M : CM_*(f) \to CM_*(f), \quad \partial_M(p) := \sum_{q \in \mathrm{Crit}(f)} n(p,q)q,$$

where $n(p,q)$ this time is the number of solutions of

$$\begin{cases} \frac{d\gamma}{dt} + \nabla f(\gamma) = 0, \\ \gamma(-\infty) = p,\ \gamma(+\infty) = q. \end{cases} \tag{4}$$

For two Morse functions f^α, f^β the isomorphism

$$T^{\alpha\beta} : HM_*(f^\alpha) \to HM_*(f^\beta)$$

is defined in a way analogous to $S^{\alpha\beta}$, by counting the numbers of the solutions of

$$\begin{cases} \frac{d\gamma}{dt} + \nabla f^{\alpha\beta}(\gamma) = 0, \\ \gamma(-\infty) = p^{\alpha},\ \gamma(+\infty) = p^{\beta} \end{cases} \tag{5}$$

(see [15] for details). Morse homology groups $HM_*(f)$ are isomorphic to singular homology groups $H_*(M;\mathbb{Z}_2)$ [7,15].

Any C^2-small Morse function $f : M \to \mathbb{R}$ can be extended to a Hamiltonian $H_f : T^*M \to \mathbb{R}$ so that the intersection points $O_M \cap \phi_1^{H_f}(O_M)$ are in one-to-one correspondence with critical points of f and the solutions of (2) are in one-to-one correspondence with the solutions of (4). Hence we have the isomorphisms

$$H_*(M;\mathbb{Z}_2) \equiv HM_*(f) \equiv HF_*(H_f).$$

This construction was generalized by Pożniak [11]. He considers the conormal bundle $\nu^*N \subset T^*M$ of *closed* submanifold $N \subset M$ and the Floer chain complex $CF_*(\nu^*N, \phi_1^H(O_M))$. In this case Floer homology is defined by counting the number of solutions of

$$\begin{cases} \frac{\partial u}{\partial s} + J(\frac{\partial u}{\partial t} - X_H(u)) = 0, \\ u(s,0) \in O_M,\ u(s,1) \in \nu^*N, \\ u(-\infty,t) = \phi_t^H((\phi_1^H)^{-1})(x),\ u(+\infty,t) = \phi_t^H((\phi_1^H)^{-1})(y), \\ x, y \in \nu^*N \cap \phi_1^H(O_M), \end{cases}$$

and is isomorphic to the singular homology of N. Kasturirangan and Oh [3] generalized this construction to *open* subsets $U \subset M$.

Note that the isomorphisms $S^{\alpha\beta}$ and $T^{\alpha\beta}$ are defined by counting the numbers of the solutions of different types of Eqs. (3) and (5). Therefore, it is not obvious whether the diagram

$$\begin{tikzcd} HF_*(H^{\alpha}) \arrow[r, "S^{\alpha\beta}"] & HF_*(H^{\beta}) \\ HM_*(f^{\alpha}) \arrow[r, "T^{\alpha\beta}"] \arrow[u] & HM_*(f^{\beta}) \arrow[u] \end{tikzcd} \tag{6}$$

commutes. Commutativity of the this diagram is implicitly used in [5,6]. Similar question in Floer homology for periodic orbits is resolved by Piunikhin, Salamon and Schwarz [10]. Instead of the isomorphisms described above, they consider the isomorphisms defined by counting the intersection numbers of spaces of perturbed holomorphic spheres and spaces of gradient trajectories. In this paper, we will carry out this construction for holomorphic discs with Lagrangian boundary conditions.

2. Holomorphic discs and gradient trajectories

Let $f : M \to \mathbb{R}$ be a Morse function, $p \in M$ its critical point, $H \in C_c^{\infty}(T^*M)$ Hamiltonian, $\varphi_t^H : T^*M \to T^*M$ its Hamiltonian flow and $x : [0,1] \to T^*M$, its Hamiltonian orbit which satisfies $x(0), x(1) \in O_M$. We will always consider only *compactly supported* Hamiltonians, such that the intersection $O_M \cap \varphi_t^H(O_M)$ is transverse. We will denote by $m_f(p)$ the Morse index of f at p and by

$\mu_H(x)$ the Maslov index of the Lagrangian path $T_{x(t)}(\varphi_t^H)(O_M)$ as defined in [12,13]. Recall that $\mu_H(x)$ is always a half-integer (see [8,12,13] for details).

It is known (see Theorem 5.1 in [8]) that if $H = \pi^* f$ and if $x(t) \equiv p$ is the constant Hamiltonian orbit corresponding to the critical point $p \in M$ of f, then $m_f(p) = \mu_H(x) + \frac{n}{2}$. Here $n = \dim M$.

Morse homology $HM_k(f)$ is graded by Morse index $k = m_f(p)$ of critical points and Floer homology $HF_k(H)$ by $k = \mu_H(x) + \frac{n}{2}$ of Hamiltonian orbits that generate it.

Let R be a fixed positive number. Let $\rho_R : [0, +\infty) \to \mathbb{R}$ be a smooth function such that

$$\rho_R(t) = \begin{cases} 1, & t \geqslant R+1, \\ 0, & t \leqslant R, \end{cases}$$

and let $\mathcal{M}(p, f; x, H)$ be the space of pairs of maps

$$\gamma : (-\infty, 0] \to M, \qquad u : [0, +\infty) \times [0, 1] \to T^*M$$

that satisfy

$$\begin{cases} \frac{d\gamma}{dt} = -\nabla f(\gamma(t)), \\ \frac{\partial u}{\partial s} + J(\frac{\partial u}{\partial t} - X_{\rho_R H}(u)) = 0, \\ u(\partial([0, +\infty) \times [0, 1])) \in O_M, \\ \gamma(-\infty) = p, u(+\infty, t) = x(t), \\ \gamma(0) = u(0, \frac{1}{2}). \end{cases} \tag{7}$$

Let $\rho_R : (-\infty, 0] \to \mathbb{R}$ be a smooth function such that

$$\rho_R(t) = \begin{cases} 1, & t \leqslant -R-1, \\ 0, & t \geqslant -R, \end{cases}$$

and let $\mathcal{M}(x, H; p, f)$ be the space of pairs of maps

$$u : (-\infty, 0] \times [0, 1] \to T^*M, \qquad \gamma : [0, +\infty) \to M$$

that satisfy

$$\begin{cases} \frac{d\gamma}{dt} = -\nabla f(\gamma(t)), \\ \frac{\partial u}{\partial s} + J(\frac{\partial u}{\partial t} - X_{\rho_R H}(u)) = 0, \\ u(\partial((-\infty, 0] \times [0, 1])) \subset O_M, \\ u(-\infty, t) = x(t), \gamma(+\infty) = p, \\ \gamma(0) = u(0, \frac{1}{2}). \end{cases} \tag{8}$$

See also Fig. 1.

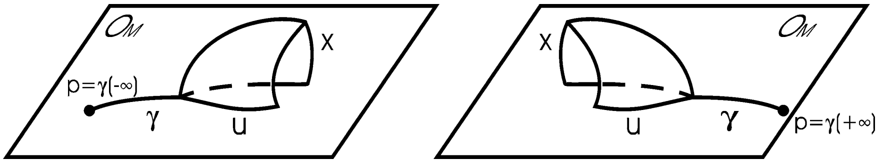

Fig. 1. $\mathcal{M}(p, f; x, H)$ and $\mathcal{M}(x, H; p, f)$.

Proposition 1. *For a generic Morse function $f:M\to\mathbb{R}$ and compactly supported Hamiltonian $H:T^*M\to\mathbb{R}$ the set $\mathcal{M}(p,f;x,H)$ is a smooth manifold of dimension $m_f(p)-(\mu_H(x)+\frac{n}{2})$ and $\mathcal{M}(x,H;p,f)$ is a smooth manifold of dimension $\mu_H(x)+\frac{n}{2}-m_f(p)$.*

Proof. Let $W^u(p,f)$ be the unstable manifold of the critical point p of a Morse function f and let $W^s(x,H)$ be the set of solutions of

$$\begin{cases} \frac{\partial u}{\partial s}+J(\frac{\partial u}{\partial t}-X_{\rho_R H}(u))=0,\\ u(\partial([0,+\infty)\times[0,1]))\in O_M,\\ u(+\infty,t)=x(t). \end{cases}$$

Then $\dim W^u(p,f)=m_f(p)$ (see [7]) and $\dim W^s(x,H)=-\mu_H(x)+\frac{n}{2}$ (see Appendix in [9]). For generic f, H the evaluation map

$$e:W^u(p,f)\times W^s(x,H)\to M\times M,\qquad (\gamma,u)\mapsto \bigl(\gamma(0),u(0,1/2)\bigr)$$

is transversal to the diagonal and therefore $\mathcal{M}(p,f;x,H)=e^{-1}(\Delta)$ is a smooth manifold of codimension n in $W^u(p,f)\times W^s(x,H)$, and hence of dimension $m_f(p)-\mu_H(x)+\frac{n}{2}-n=m_f(p)-(\mu_H(x)+\frac{n}{2})$. The proof for $\mathcal{M}(x,H;p,f)$ is analogous. □

The proof of the following proposition is standard, and follows from the Gromov compactness and the Arzela–Ascoli theorem.

Proposition 2. *If f and H are as in Proposition 1 and $m_f(p)=\mu_H(x)+\frac{n}{2}$ then $\mathcal{M}(p,f;x,H)$ and $\mathcal{M}(x,H;p,f)$ are finite sets.*

Definition 3. If $m_f(p)=\mu_H(x)+\frac{n}{2}$ we denote the $(\mathrm{mod}\,\mathbb{Z}_2)$ cardinal numbers of $\mathcal{M}(p,f;x,H)$ and $\mathcal{M}(x,H;p,f)$ by $n(p,f;x,H)$ and $n(x,H;p,f)$ respectively.

3. Isomorphism

Define homomorphisms

$$\phi:CF_k(H)\to CM_k(f),\qquad \psi:CM_k(f)\to CF_k(H)$$

by

$$x\mapsto\sum_{m_f(p)=k}n(x,H;p,f)p,\qquad p\mapsto\sum_{\mu_H(x)+n/2=k}n(p,f;x,H)x$$

on the generators.

Proposition 4. *Homomorphisms ϕ and ψ are well defined and satisfy*

$$\phi\circ\partial_F=\partial_M\circ\phi,\qquad \psi\circ\partial_M=\partial_F\circ\psi.$$

Proof. That ϕ and ψ are well defined follows from the definition of grading of Morse and Floer homology and Proposition 2.

Let $\mathcal{M}(x,y,H)$ be the set of solution of (2) and let $\mathcal{M}(p,q,f)$ be the set of solution of (4) (modulo $\mathbb{R}$-action). Assume that $\mu_H(y)+\frac{n}{2}=k-1$, $\mu_H(x)+\frac{n}{2}=k$, $m_f(p)=k$, $m_f(q)=k-1$.

From compactness and gluing arguments (see, e.g., [4,15,16]) and from the dimensional reasons (Proposition 1) it follows that

$$\partial\mathcal{M}(x,H;q,f)=\bigcup_{\mu_H(y)+n/2=k-1}\mathcal{M}(x,y,H)\times\mathcal{M}(y,H;q,f)\\ \cup\bigcup_{m_f(p)=k}\mathcal{M}(x,H;p,f)\times\mathcal{M}(p,q,f).$$

This means that the number of broken trajectories in $\mathcal{M}(x,y,H)\times\mathcal{M}(y,H;p,f)\cup\mathcal{M}(p,q,f)\times\mathcal{M}(q,f;x,H)$ which define the difference $\phi\circ\partial_F-\partial_M\circ\phi$ is zero, as the number of ends of dim-one manifold $\partial\mathcal{M}(x,H;q,f)$ mod $\mathbb{Z}_2$:

$$(\phi\circ\partial_F-\partial_M\circ\phi)(x)=\sum_{m_f(q)=k-1}\Biggl(\sum_{\mu_H(y)+n/2=k-1}n(x,y,H)n(y,H;q,f)\Biggr)q\\ -\sum_{m_f(q)=k-1}\Biggl(\sum_{m_f(p)=k}n(x,H;p,f)n(p,q,f)\Biggr)q=0.$$

Hence $\phi\circ\partial_F=\partial_M\circ\phi$, and by the same argument $\psi\circ\partial_M=\partial_F\circ\psi$. □

Corollary 5. *Chain homomorphisms ϕ and ψ induce homomorphisms*

$$\Phi: HF_k(H)\to HM_k(f),\qquad \Psi: HM_k(f)\to HF_k(H)$$

in homology.

Theorem 6. *Homomorphisms Φ and Ψ are isomorphisms and*

$$\Phi\circ\Psi=Id,\qquad \Psi\circ\Phi=Id. \tag{9}$$

Proof. That Φ and Ψ are isomorphisms will follow from (9). Let us prove (9). By definition $\phi\circ\psi: CM_k(f)\to CM_k(f)$ takes the value

$$\phi\circ\psi(p)=\sum_{m_f(q)=k}\Biggl(\sum_{\mu_H(x)+n/2=k}n(p,f;x,H)n(x,H;q,f)\Biggr)q$$

on the generator p of $CM_k(f)$ (Fig. 2). Note that $\sum_x n(p,f;x,H)n(x,H;q,f)$ is the number of points in the set

$$\mathcal{M}(p,f;x,H)\times\mathcal{M}(x,H;q,f),$$

which is, as we sketch below, by compactness and gluing arguments, the boundary component of the set

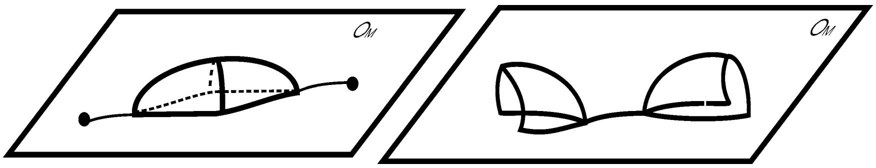

Fig. 2. $\phi \circ \psi$ and $\psi \circ \phi$.

$$\overline{\mathcal{M}}(p, q, f; H) := \left\{ (\gamma_-, \gamma_+, u, R) \;\middle|\; \begin{array}{l} \gamma_- : (-\infty, 0] \to M,\ \gamma_+ : [0, +\infty) \to M, \\ u : \mathbb{R} \times [0,1] \to T^*M, \\ \frac{d\gamma_\pm}{dt} = -\nabla f(\gamma_\pm), \\ \frac{\partial u}{\partial s} + J(\frac{\partial u}{\partial t} - X_{\rho_R H}(u)) = 0, \\ \gamma_-(-\infty) = p,\ \gamma_+(+\infty) = q, \\ u(\partial(\mathbb{R} \times [0,1])) \subset O_M, \\ u(\pm\infty, t) = \gamma_\pm(0) \end{array} \right\},$$

where $\rho_R : \mathbb{R} \to [0,1]$ is a smooth function such that

$$\rho_R(t) = \begin{cases} 1, & |t| \leqslant R, \\ 0, & |t| \geqslant R + 1 \end{cases}$$

for $R > 0$. Moreover, by argument similar to [14,16] it can be proved that there exists $R_0 > 0$ such that there exists a *gluing map*

$$\imath : \mathcal{M}(p, f; x, H) \times \mathcal{M}(x, H; q, f) \times (-\infty, R_0] \to \overline{\mathcal{M}}(p, q, f; H).$$

By Arzela–Ascoli theorem and Gromov compactness and gluing it follows that the boundary of one-dimensional manifold $\overline{\mathcal{M}}(p, q, f; H)$ is the union of $\mathcal{M}(p, f; x, H) \times \mathcal{M}(x, H; q, f)$, $\imath(\mathcal{M}(p, f; x, H) \times \mathcal{M}(x, H; q, f) \times \{R_0\})$ and the broken trajectories in $\mathcal{M}(p, r, f) \times \overline{\mathcal{M}}(r, q, f; H)$ and $\overline{\mathcal{M}}(p, r, f; H) \times \mathcal{M}(r, q, f)$.

More precisely, from Arzela–Ascoli theorem and Gromov compactness it follows that the sequence $(\gamma_-^n, \gamma_+^n, u_n, R_n) \in \overline{\mathcal{M}}(p, q, f; H)$ has a C^∞_{loc}-convergent subsequence. If it does not have a subsequence which converges in $\overline{\mathcal{M}}(p, q, f; H)$, then one of the following four statements must hold:

(1) There is some subsequence that converges to an element in $\mathcal{M}(p, r, f) \times \overline{\mathcal{M}}(r, q, f; H)$.
(2) There is a subsequence that converges to an element in $\overline{\mathcal{M}}(p, r, f; H) \times \mathcal{M}(r, q, f)$.
(3) There is a subsequence that converges to an element in $\mathcal{M}(p, q, f; H)$, the set of all (γ_-, γ_+, u) such that $\gamma_- : (-\infty, 0] \to M$, $\gamma_+ : [0, +\infty) \to M$, $u : \mathbb{R} \times [0,1] \to T^*M$ are the solutions of $\frac{d\gamma_\pm}{dt} = \nabla f(\gamma_\pm)$, $\frac{\partial u}{\partial s} + J(\frac{\partial u}{\partial t} - X_{\rho_{R_0}}(u)) = 0$ and $u(\pm\infty, t) = \gamma_\pm(0)$.
(4) There is a subsequence that converges to an element in $\mathcal{M}(p, f; x, H) \times \mathcal{M}(x, H; q, f)$.

First two cases correspond to $\partial_M \circ K$ and $K \circ \partial_M$. In this cases the subsequence of R_n is convergent. In the third case $R_n \to R_0$. If $R_n \to \infty$, then the reparametrisations $\hat{u}_n := u_n(s - R_n - R_0 - 1, t)$, $\check{u}_n := u_n(s + R_n + R_0 + 1, t)$ give rise to the pair of sequences $(\hat{u}_n, \check{u}_n)$ which converges to the pair $(\hat{u}, \check{u})$ that can be identified with the pair that satisfies the same equations on $[0, +\infty) \times [0,1]$ and $(-\infty, 0] \times [0,1]$ such that $\hat{u}(0, 1/2) = \gamma_-(0)$, $\check{u}(0, 1/2) = \gamma_+(0)$.

Conversely, to each broken object mentioned above, i.e., $(\gamma_-^1, \gamma_-^2, \gamma_+, u, R)$ in $\mathcal{M}(p, r, f) \times \overline{\mathcal{M}}(q, f; H)$, $(\gamma_-, \gamma_+^1, \gamma_+^2, u, R)$ in $\overline{\mathcal{M}}(p, r, f; H) \times \mathcal{M}(r, q, f)$, $(\gamma_-, \gamma_+, u, R_0)$ in $\mathcal{M}(p, q, f; H)$,

$(\gamma_+, \gamma_-, u_-, u_+)$ in $\mathcal{M}(p, f; x, H) \times \mathcal{M}(x, H; q, f)$, there corresponds an element in $\overline{\mathcal{M}}(p, q, f, H)$. Gluing arguments for cases (1) and (2) are completely analogous to Theorem 5 of Section 2.5.2 in [15], and for case (4) to Section 3.3 in [14]. Consider homomorphisms

$$F : CM_k(f) \to CM_k(f), \quad p \mapsto \sum_{m_f(q)=k} n(p, q, f, H)q, \tag{10}$$

where $n(p, q, f, H)$ is the number of intersections of the space of perturbed holomorphic discs with the boundary on O_M with the unstable manifold $W^u(p, f)$ and the stable manifold $W^s(q, f)$, and

$$K : CM_k(f) \to CM_{k+1}(f), \quad p \mapsto \sum_{m_f(q)=k+1} \bar{n}(p, q, f, H)q,$$

where $\bar{n}(p, q, f, H)$ is the number of points in $\overline{\mathcal{M}}(p, q, f; H)$. Then $\phi \circ \psi - F = \partial_M \circ K + K \circ \partial_M$, i.e., $\phi \circ \psi$ is chain homotopic to F. The homomorphism

$$HM_k(f) \to HM_k(f)$$

induced by the chain map (10) is independent of the choice of Hamiltonian H. Indeed, let H_0, H_1 be two Hamiltonians, H_λ, $0 \leqslant \lambda \leqslant 1$, the homotopy between them, and F_0, F_1 the chain homomorphisms that correspond to H_0, H_1. Consider the space

$$\mathcal{M}_\lambda(p, q, f; H_\lambda) := \bigl\{(u, \gamma_1, \gamma_2, \lambda) \mid (u, \gamma_1, \gamma_2) \in \mathcal{M}(p, q, f; H_\lambda)\bigr\}.$$

Its dimension is $m_f(p) - m_f(q) + 1 = 1$, and its boundary is

$$\begin{aligned} \partial\mathcal{M}_\lambda(p, q, f; H_\lambda) = {} & \mathcal{M}(p, q, f; H_0) - \mathcal{M}(p, q, f; H_1) \\ & + \bigcup_{m_f(r)=k-1} \mathcal{M}(p, r; f) \times \mathcal{M}_\lambda(r, q, f; H_\lambda) \\ & + \bigcup_{m_f(r)=k+1} \mathcal{M}_\lambda(p, r, f; H_\lambda) \times \mathcal{M}(r, q; f). \end{aligned}$$

It follows that $F_1 - F_0 = \partial_M \circ K + K \circ \partial_M$ for

$$K : CM_k(f) \to CM_{k+1}(f), \quad K(p) = \sum_{m_f(r)=k+1} n(p, r, f; H_\lambda)r,$$

i.e., F_0 and F_1 are chain homotopic. Indeed, if

$$\begin{aligned} n_q := {} & n(p, q, f; H_0) - n(p, q, f; H_1) + \sum_{m_f(r)=k-1} n(p, r, f)n(r, q, f; H_\lambda) \\ & + \sum_{m_f(r)=k+1} n(p, r, f; H_\lambda)n(r, q, f), \end{aligned}$$

then $n_q = 0$ since this is the number of boundary points of one-dimensional manifold $\mathcal{M}_\lambda(p, q, f; H_\lambda)$, and therefore

$$F_0(p) - F_1(p) + \partial_M \circ K(p) + K \circ \partial_M(p) = \sum_{m_f(q)=m_f(p)} n_q q = 0.$$

Chose now the homotopy between H and 0 we see that the map (10) is chain homotopic to the map

$$p \mapsto \sum_{m_f(q)=m_f(p)} n(p,q,f,0)q.$$

But $n(p,q,f,0)$ is the number of intersections of the space of *unperturbed* holomorphic discs with the boundary on O_M with the unstable manifold $W^u(p,f)$ and the stable manifold $W^s(q,f)$. Since every holomorphic discs with the boundary on O_M is constant, $n(p,q,f,0)$ is just the number of points in $W^u(p,f)\cap W^s(q,f)$. But since $m_f(p)=m_f(q)$, we have

$$W^u(p,f)\cap W^s(q,f) = \begin{cases} \{p\}, & p=q, \\ \emptyset, & p\neq q, \end{cases}$$

and hence

$$n(p,q,f,0) = \begin{cases} 1, & p=q, \\ 0, & p\neq q, \end{cases}$$

and thus (10) is chain homotopic to the identity.

Proof of the second statement in (9) is similar. Again, by gluing and compactness arguments, the chain homomorphism $\psi\circ\phi : CF_k(H)\to CF_k(H)$ is chain homotopic to the homomorphism

$$G : x \mapsto \sum_{\mu_H(y)=\mu_H(x)} n(\varepsilon; x, y, H; f)y,$$

where $n(\varepsilon; x, y, H; f)y$ is the number of points of the space

$$\mathcal{M}(\varepsilon; x, y, H; f) := \left\{ (u_-, u_+, \gamma) \;\middle|\; \begin{array}{l} u_- : (-\infty, 0]\times[0,1]\to T^*M, \\ u_+ : [0,+\infty)\times[0,1]\to T^*M, \\ \gamma : [-\varepsilon,\varepsilon]\to M, \\ \frac{d\gamma}{dt} = -\nabla f(\gamma), \\ \frac{\partial u_\pm}{\partial s} + J(\frac{\partial u_\pm}{\partial s} - X_{\rho_R H}(u_\pm)) = 0, \\ u_-(\partial((-\infty,0]\times[0,1]))\subset O_M, \\ u_+(\partial([0,+\infty)\times[0,1]))\subset O_M, \\ u_\pm(0,1/2) = \gamma(\pm\varepsilon), \\ u_-(-\infty,t) = x(t),\ u_+(+\infty,t) = y(t) \end{array} \right\}.$$

The induced homomorphism in homology for different ε are chain homotopic. Indeed, let G_0, G_1 be two chain homomorphisms corresponding to two values $\varepsilon_0, \varepsilon_1$ and for $\varepsilon\in[\varepsilon_0,\varepsilon_1]$ let

$$\mathcal{M}_\varepsilon(x,y,H;f) := \big\{(u_-,u_+,\gamma,\varepsilon) \mid (u_-,u_+,\gamma)\in\mathcal{M}(\varepsilon;x,y,H;f)\big\}.$$

If $\mu_H(y) = \mu_H(x)+1$ denote by $n_\varepsilon(x,y;H,f)$ be the number of points in the zero-dimensional manifold $\mathcal{M}_\varepsilon(x,y,H;f)$ and let

$$K : CF_k(H)\to CF_{k+1}(H), \quad x\mapsto \sum_{\mu_H(y)+n/2=k+1} n_\varepsilon(x,y,H;f)y.$$

Then from

$$\begin{aligned}\partial\mathcal{M}_\varepsilon(x,y,H;f) = {} & \mathcal{M}(\varepsilon_1;x,y,H;f) - \mathcal{M}(\varepsilon_0;x,y,H;f) \\ & + \bigcup_{\mu_H(z)=\mu_H(x)-1} \mathcal{M}(x,z,H)\times\mathcal{M}_\varepsilon(z,y,H;f) \\ & + \bigcup_{\mu_H(z)=\mu_H(x)+1} \partial\mathcal{M}_\varepsilon(x,z,H;f)\times\mathcal{M}(z,y,H)\end{aligned}$$

follows that $G_0 - G_1 = K\circ\partial_F + \partial_F\circ K$.

By letting $\varepsilon\to 0$ we get that $\psi\circ\phi$ is chain homotopic to homomorphism

$$x \mapsto \sum_{\mu_H(y)=\mu_H(x)} \tilde{n}(x,y;H),$$

where $\tilde{n}(x,y;H)$ is the number of points in the set

$$\widetilde{\mathcal{M}}(x,y,H) := \left\{ (u_-,u_+) \;\middle|\; \begin{array}{l} u_- : (-\infty,0]\times[0,1]\to T^*M, \\ u_+ : [0,+\infty)\times[0,1]\to T^*M, \\ \frac{\partial u}{\partial s} + J(\frac{\partial u}{\partial t} - X_{\rho_R H}(u)) = 0, \\ u_-(\partial((-\infty,0]\times[0,1])) \subset O_M, \\ u_+(\partial([0,+\infty)\times[0,1])) \subset O_M, \\ u_-(0,1/2) = u_+(0,1/2), \\ u_-(-\infty,t) = x(t),\ u_+(+\infty,t) = y(t) \end{array} \right\}.$$

Now again, by gluing and compactness, we conclude that $\psi\circ\phi$ is chain homotopic to

$$x \mapsto \sum_{\mu_H(y)=\mu_H(x)} n(x,y;H)y, \tag{11}$$

where $n(x,y;H)$ is the number of solutions of (2). Here, in a similar way as before, we consider the one-dimensional manifold

$$\overline{\mathcal{M}}(x,y,H) := \left\{ (R,u) \;\middle|\; \begin{array}{l} u : \mathbb{R}\times[0,1]\to T^*M, \\ \frac{\partial u}{\partial s} + J(\frac{\partial u}{\partial t} - X_{\rho_R H}(u)) = 0, \\ u(\partial(\mathbb{R}\times[0,1])) \subset O_M, \\ u(-\infty,t) = x(t),\ u(+\infty,t) = y(t) \end{array} \right\},$$

and identify its boundary with

$$\mathcal{M}_{R_0}(x,y,H) \cup \bigcup_z \mathcal{M}(x,z,H)\times\overline{\mathcal{M}}(z,y,H) \cup \bigcup_z \overline{\mathcal{M}}(x,z,H)\times\mathcal{M}(z,y,H) \cup \widetilde{\mathcal{M}}(x,y,H).$$

Here

$$\mathcal{M}_{R_0}(x,y,H) := \left\{ u : \mathbb{R}\times[0,1]\to T^*M \;\middle|\; \begin{array}{l} \frac{\partial u}{\partial s} + J(\frac{\partial u}{\partial t} - X_{\rho_{R_0} H}(u)) = 0, \\ u(\partial(\mathbb{R}\times[0,1])) \subset O_M, \\ u(-\infty,t) = x(t),\ u(+\infty,t) = y(t) \end{array} \right\},$$

and the elements from $\widetilde{\mathcal{M}}(x,y,H)$ come as the boundary elements after the same reparametrisation as in the proof of Theorem 6. Since $\mu_H(x) = \mu_H(y)$, by dimensional reasons,

$$n(x,y;H) = \begin{cases} 1, & x = y, \\ 0, & x \neq y, \end{cases}$$

and thus homomorphism (11) is the identity. □

4. Commutative diagram

Consider the chain homomorphisms

$$\psi^\alpha : CM_k(f^\alpha) \to CF_k(H^\alpha), \qquad \phi^\beta : CF_k(H^\beta) \to CM_k(f^\beta),$$
$$\sigma^{\alpha\beta} : CF_k(H^\alpha) \to CF_k(H^\beta), \qquad \tau^{\alpha\beta} : CM_k(f^\alpha) \to CM_k(f^\beta)$$

introduced above. They induce the isomorphisms

$$\Psi^\alpha : HM_k(f^\alpha) \to HF_k(H^\alpha), \qquad \Phi^\beta : HF_k(H^\beta) \to HM_k(f^\beta),$$
$$S^{\alpha\beta} : HF_k(H^\alpha) \to HF_k(H^\beta), \qquad T^{\alpha\beta} : HM_k(f^\alpha) \to HM_k(f^\beta).$$

Theorem 7. *Diagram*

$$\begin{tikzcd} HF_*(H^\alpha) \arrow[r, "S^{\alpha\beta}"] & HF_*(H^\beta) \\ HM_*(f^\alpha) \arrow[u, "\Psi^\alpha"] \arrow[r, "T^{\alpha\beta}"] & HM_*(f^\beta) \arrow[u, "\Psi^\beta=(\Phi^\beta)^{-1}"'] \end{tikzcd} \tag{12}$$

commutes.

Proof. We want to prove that

$$S^{\alpha\beta} \circ \Psi^\alpha = \Psi^\beta \circ T^{\alpha\beta}. \tag{13}$$

Let $f^{\alpha\beta}$ be the homotopy connecting f^α and f^β and let $H^{\alpha\beta}$ be the homotopy connecting H^α and H^β. Chain homomorphism $\sigma^{\alpha\beta} \circ \psi^\alpha$ is obtained by counting the numbers of points in the sets

$$\mathcal{M}(p^\alpha, f^\alpha; x^\alpha, H^\alpha) \times \mathcal{M}(x^\alpha, x^\beta, H^{\alpha\beta}),$$

where $\mathcal{M}(x^\alpha, x^\beta, H^{\alpha\beta})$ is the set of the solutions of (3), and chain homomorphism $\phi^\beta \circ \tau^{\alpha\beta}$ by counting the numbers of points in the sets

$$\mathcal{M}(p^\alpha, p^\beta, f^{\alpha\beta}) \times \mathcal{M}(p^\beta, f^\beta; x^\beta, H^\beta),$$

where $\mathcal{M}(p^\alpha, p^\beta, f^{\alpha\beta})$ is the set of the solutions of (5), where $m_{f^\alpha}(p^\alpha) = n/2 + \mu_{H^{\alpha\beta}}(x^\alpha) = n/2 + \mu_{H^{\alpha\beta}}(x^\beta) = k$. By gluing and compactness argument similar to the ones used before, $\sigma^{\alpha\beta} \circ \psi^\alpha$ and $\psi^\beta \circ \tau^{\alpha\beta}$ are chain homotopic to the chain homomorphisms ϑ and φ obtained by counting the numbers of points in $\mathcal{M}(p^\alpha, f^\alpha; x^\beta, H^{\alpha\beta})$ and $\mathcal{M}(p^\alpha, f^{\alpha\beta}; x^\beta, H^\beta)$ respectively. Let $(f_\lambda^{\alpha\beta}, H_\lambda^{\alpha\beta})_{0\leqslant\lambda\leqslant 1}$ be a homotopy connecting $(f_\lambda^{\alpha\beta}, H_\lambda^{\alpha\beta})|_{\lambda=0} = (f^\alpha, H^{\alpha\beta})$ and $(f_\lambda^{\alpha\beta}, H_\lambda^{\alpha\beta})|_{\lambda=1} = (f^{\alpha\beta}, H^\beta)$. Consider the set

$$\mathcal{M}_\lambda(p^\alpha, f_\lambda^{\alpha\beta}; x^\beta, H_\lambda^{\alpha\beta}) := \left\{ (\gamma, u, \lambda) \;\middle|\; \begin{array}{l} \gamma : (-\infty, 0] \to M, \\ u : [0, +\infty) \times [0, 1] \to T^*M, \\ \frac{d\gamma}{dt} = -\nabla f_\lambda^{\alpha\beta}(\gamma), \\ \frac{\partial u}{\partial s} + J(\frac{\partial u}{\partial t} - X_{\rho_R H_\lambda^{\alpha\beta}}(u)) = 0, \\ u(\partial([0, +\infty) \times [0, 1])) \subset O_M, \\ \gamma(0) = u(0, 1/2) \end{array} \right\}.$$

Then $\dim \mathcal{M}_\lambda(p^\alpha, f_\lambda^{\alpha\beta}; x^\beta, H_\lambda^{\alpha\beta}) = m_{f^{\alpha\beta}}(p^\alpha) - (\mu_{H^{\alpha\beta}}(x^\beta) + n/2) + 1$. If $m_{f^{\alpha\beta}}(p^\alpha) = \mu_{H^{\alpha\beta}}(x^\beta) + n/2$ then the boundary of one-dimensional manifold $\mathcal{M}(p^\alpha, f_\lambda^{\alpha\beta}; x^\beta, H_\lambda^{\alpha\beta})$ is

$$\begin{aligned}\partial\mathcal{M}_\lambda(p^\alpha, f_\lambda^{\alpha\beta}; x^\beta, H_\lambda^{\alpha\beta}) = &\bigcup_{m_{f^\alpha}(r^\alpha)=m_{f^\alpha}(p^\alpha)-1} \mathcal{M}(p^\alpha, r^\alpha; f^\alpha) \times \mathcal{M}_\lambda(r^\alpha, f_\lambda^{\alpha\beta}; x^\beta, H_\lambda^{\alpha\beta}) \\ &+ \bigcup_{\mu_{H^{\alpha\beta}}(y^\beta)=\mu_{H^{\alpha\beta}}(x^\beta)+1} \mathcal{M}_\lambda(p^\alpha, f_\lambda^{\alpha\beta}; y^\beta, H_\lambda^{\alpha\beta}) \times \mathcal{M}(y^\beta, x^\beta, H^\beta) \\ &+ \mathcal{M}(p^\alpha, f^{\alpha\beta}; x^\beta, H^\beta) - \mathcal{M}(p^\alpha, f^\alpha; x^\beta, H^{\alpha\beta}).\end{aligned}$$

The chain map obtained by counting boundary components of $\mathcal{M}_\lambda(p^\alpha, f_\lambda^{\alpha\beta}; x^\beta, H_\lambda^{\alpha\beta})$ defines the chain homomorphism $\vartheta - \varphi + K \circ \partial_M - \partial_F \circ K$ for the chain homomorphism $K : CM_{k-1}(f^\alpha) \to CF_k(H^\beta)$ defined by

$$K(p^\alpha) := \sum_{\mu_{H^\beta}(x^\beta)+n/2=k} n(p^\alpha, f_\lambda^{\alpha\beta}; x^\beta, H_\lambda^{\alpha\beta}) x^\beta.$$

Therefore, ϑ and φ (and hence $\sigma^{\alpha\beta} \circ \psi^\alpha$ and $\phi^\beta \circ \tau^{\alpha\beta}$) are chain homotopic. This proves (13). □

Acknowledgements

We would like to thank M. Schwarz for several discussions about the paper [10]. We would also like to thank the referee for several useful remarks and suggestions. This work is partially supported by Serbian Ministry of Science, Technology and Development project #1643.

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